Problem 1) a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$

To determine A, B, and C, we combine the three fractions and set the numerator equal to 1, that is,

$$A(n+1)(n+2) + Bn(n+2) + Cn(n+1) = n^2(A+B+C) + n(3A+2B+C) + 2A = 1$$

$$\Rightarrow A = 1/2, \quad B+C = -1/2, \quad 2B+C = -3/2 \Rightarrow B = -1, \quad C = 1/2.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n+1} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+2}$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \cancel{\sum_{n=3}^{\infty}\frac{1}{n}}\right) - \left(\frac{1}{2} + \cancel{\sum_{n=3}^{\infty}\frac{1}{n}}\right) + \frac{1}{2}\cancel{\sum_{n=3}^{\infty}\frac{1}{n}}$$

$$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

b) We now use a somewhat different method of calculating the partial fractions. Writing

$$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3},$$

We multiply both sides of the above equation with x, then set $x=0$ to find

$$\frac{\cancel{x}}{\cancel{x}(x+1)(x+2)(x+3)} = A + \frac{Bx}{x+1} + \frac{Cx}{x+2} + \frac{Dx}{x+3} \xRightarrow{x=0} A = \frac{1}{6}$$

To find B, we multiply by $x+1$, then set $x=-1$, and so on. We'll have:

$B = -\frac{1}{2}$, $C = \frac{1}{2}$, $D = -\frac{1}{6}$. Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+1} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+2} - \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n+3}$$

$$= \frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{3} + \cancel{\sum_{n=4}^{\infty}\frac{1}{n}}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \cancel{\sum_{n=4}^{\infty}\frac{1}{n}}\right) + \frac{1}{2}\left(\frac{1}{3} + \cancel{\sum_{n=4}^{\infty}\frac{1}{n}}\right) - \frac{1}{6}\cancel{\sum_{n=4}^{\infty}\frac{1}{n}} = \frac{1}{18}$$

c) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}\left(1+\cancel{\frac{1}{3}}+\cancel{\frac{1}{5}}+\cdots\right)-\frac{1}{2}\left(\cancel{\frac{1}{3}}+\cancel{\frac{1}{5}}+\cancel{\frac{1}{7}}\cdots\right)$

$= \frac{1}{2}$